1. *Interpret the iterated integral* $\int_0^1 \int_{x^2}^x (x^2+y^2)\ dy\ dx$ *as an integral over a certain Jordan region in* $\mathbb{R}^2$ *and compute this integral.*

   Solution: $\int_0^1 \int_{x^2}^x (x^2+y^2)\ dy\ dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^x\ dx = \int_0^1 \frac{4}{3}x^3 - x^4 - \frac{x^6}{3}\ dx = \frac{3}{35}$. $\square$

2. *Let* $A = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2+y^2 \leq 4, x^2-y^2 \geq 1\}$*. Compute* $\int_A xy\ dV(x,y)$*, by making a change of variables* $u = x^2+y^2, v = x^2-y^2$ *for* $x \geq 0, y \geq 0$*. State the theorem which is used.*

   Solution: We use the change of variable theorem. This is as follows: Suppose T is a $1-1$ C^1-mapping of an open set $E \subseteq \mathbb{R}^k$ into $\mathbb{R}^k$ such that the Jacobian $J_T(\mathbf{x}) \neq 0$ for all $x \in E$. If f is a continuous function on $\mathbb{R}^k$ whose support is compact and lies in $T(E)$, then

   $$\int_{\mathbb{R}^k} f(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^k} f(T(\mathbf{x}))|J_T(\mathbf{x})|d\mathbf{x}.$$

   We are given $u = x^2+y^2, v = x^2-y^2$ for $x \geq 0, y \geq 0$. Hence $T(u,v) = (\sqrt{\frac{u+v}{2}}, \sqrt{\frac{u-v}{2}})$. Thus $|J_T(u,v)| = \frac{1}{2^{\frac{1}{2}}}(u^2-v^2)^{-\frac{1}{2}}$. Hence $\int_A xy\ dV(x,y) = \frac{1}{2^{\frac{3}{2}}}\int_{T^{-1}(A)}(u^2-v^2)^{\frac{1}{2}}(u^2-v^2)^{-\frac{1}{2}}dudv = \frac{1}{2^{\frac{3}{2}}}V(T^{-1}(A))$. It is easy to see that the region A is mapped under T^{-1} to the triangle in the $u-v$ plane given by: $u \leq 4, v \geq 1, u = v$. Hence the required integral is equal to $\frac{9}{2^{\frac{5}{2}}}$. $\square$

3. *Let* C *be the curve obtained by intersecting (in* $\mathbb{R}^3$*) the plane* $x = z$ *with the cylinder* $x^2+y^2 = 1$*, oriented anticlockwise when viewed from above (positive* z*-axis). Let* S *be the region inside this curve, oriented with the upward pointing normal. Let* $F = (x, z, 2y)$ *be the component vector field of a* 1*-form on* $\mathbb{R}^3$*. Compute both sides of Stoke's theorem in this situation, and verify they are equal.*

   Solution: Stoke's theorem states the following: If F is a vector field of class C^1 in an open set $\mathbb{R}^3$, and if S is a 2-surface of class C^2 in E, then

   $$\int_S (\nabla \times F)\cdot n\, dA = \int_{\partial S}(F\cdot t)\, ds.$$

   Given that $F = (x,z,2y)$, we see that $\nabla \times F = (1,0,0)$. Here S is given by $z^2+y^2 = 1$. In polar coordinates, $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r\cos\theta)$. Hence $\frac{\partial\Phi}{\partial r} \times \frac{\partial\Phi}{\partial\theta} = -r(1,0,-1)$. Thus $\int_\Phi(\nabla\times F)\cdot n\, dA = \int_0^{2\pi}\int_0^1 -rdrd\theta = -\pi$. On the other hand, the boundary is given by $(\cos\theta, \sin\theta, \cos\theta)$. Thus $\int_{\partial S}(F\cdot t)\, ds = \int_{\partial S} xdx + zdy + 2ydz = \int_0^{2\pi} -\cos\theta\sin\theta + \cos^2\theta - 2\sin^2\theta d\theta = -\pi$. $\square$

4. *Let* $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$ *be the solid ball of radius less than or equal to* 1 *in* $\mathbb{R}^3$*. Use Gauss' theorem to compute the integral of the* 2*-form* $(x-\cos y)dy\wedge dz + (xe^z - y)dz\wedge dx + (z - \sin(2xy))dx\wedge dy$ *over the boundary of* B *(which is the sphere* S^2 *of radius* 1*, with the normal vector assumed to be pointing outward at each point).*

   Solution: Let $\omega = (x-\cos y)dy\wedge dz + (xe^z-y)dz\wedge dx + (z-\sin(2xy))dx\wedge dy$. Then $d\omega = \frac{\partial(x-\cos y)}{\partial x}dy\wedge dz\wedge dx + \frac{\partial(xe^z-y)}{\partial y}dz\wedge dx\wedge dy + \frac{\partial(z-\sin(2xy))}{\partial z}dx\wedge dy\wedge dz = dx\wedge dy\wedge dz$. Hence by Gauss' theorem $\int_{\partial B}\omega = \int_B d\omega = \frac{4\pi}{3}$. $\square$

5. *For $n \in \mathbb{N}$, define $f_n(x) = \frac{x}{1+nx^2}$, for any $x \in \mathbb{R}$. Does the sequence of functions $\{f_n\}$ converge uniformly on $\mathbb{R}$?*

   Solution: The given sequence $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to 0. To see this, observe that $|f_n(x)| = \frac{1}{\sqrt{n}}(\frac{\sqrt{n}|x|}{1+nx^2}) = \frac{1}{\sqrt{n}}(\frac{t}{1+t^2})$, where $t = \sqrt{n}|x|$. Hence $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$ for all $x \in \mathbb{R}$, and thus converges uniformly to 0. $\square$

References

[1] Rudin, Walter."Principles of Mathematical Analysis". International Series in Pure and Applied Mathematics. (1976).